Session 1 : solutions

Exercise 1

Recall that

$$m = \mathcal{L}\left(\beta\left(J\frac{z}{2}m + h\right)\right) \qquad \mathcal{L}(x) = \coth(x) - \frac{1}{x}$$

We are interested in $T \to T_c$ where $m \to 0$

$$\left.\frac{dm}{dh}\right|_{h=0} = \underbrace{\mathcal{L}'\left(\beta J\frac{z}{2}m\right)}_{\text{at } T_c,\ \cancel{m} \to 0} \cdot \beta\left(J\frac{z}{2}\frac{dm}{dh} + 1\right)$$

$\Rightarrow \mathcal{L}'(0)$

$$\mathcal{L}(x) \simeq \frac{1}{3}x \quad \text{if} \quad x \to 0$$

$$\mathcal{L}'(0) = \frac{1}{3}$$

Recall that $\quad T_c = \frac{1}{3}\frac{Jz}{2k_B}$

$$\Rightarrow \quad \left.\frac{dm}{dh}\right|_{h=0} = \frac{T_c}{T}\left.\frac{dm}{dh}\right|_{h=0} + \frac{1}{k_B T_c}$$

$T \simeq T_c$

Then $\quad \left.\frac{dm}{dh}\right|_{h=0}\left(1 - \frac{T_c}{T}\right) = \frac{1}{k_B T_c}$

$$\Rightarrow \quad \left.\frac{dm}{dh}\right|_{h=0} = \frac{1}{k_B}(T - T_c)^{-1}$$

Exercise 2:

$$H = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - h \sum_i (\hat{h} \cdot \vec{S}_i)$$

$\hat{h} \rightarrow$ unitary vector

Then

total $\rightarrow$ $$M = \frac{1}{Z} \sum_{\{\vec{S}_i\}} \left(\sum_i \hat{h} \vec{S}_i \right) e^{\beta J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j + \beta h \sum_i (\vec{S}_i \cdot \hat{h})}$$

$$\frac{dM}{dh} = -\frac{1}{Z^2} \frac{dZ}{dh} \sum_{\{\vec{S}_i\}} \left(\sum_i \hat{h} \cdot \vec{S}_i \right) e^{-\beta H} +$$

$$+ \frac{1}{Z} \sum_{\{\vec{S}_i\}} \left(\sum_i \hat{h} \cdot \vec{S}_i \right) \left(\beta \sum_i \hat{h} \cdot \vec{S}_i \right) e^{-\beta H} =$$

$$= \beta \left\langle \left(\sum_i \hat{h} \cdot \vec{S}_i \right)^2 \right\rangle - \beta \left\langle \left(\sum_i \hat{h} \cdot \vec{S}_i \right) \right\rangle^2 =$$

$$= \underbrace{\beta \left[\left\langle \left(\sum_i \hat{h} \cdot \vec{S}_i \right)^2 \right\rangle - \left\langle \sum_i \hat{h} \cdot \vec{S}_i \right\rangle^2 \right]}_{\text{This is the variance (fluctuations) of the magnetization}}$$

Exercise 3 :

$$U = \frac{1}{Z} \sum_{\{\vec{s}\}} H(\{\vec{s}\})\, e^{-\beta H(\{\vec{s}\})}$$

$$\frac{dU}{dT} = -\frac{1}{Z^2}\frac{dZ}{dT} \sum_{\{\vec{s}\}} H e^{-\beta H} +$$

$$+ \frac{1}{k_B T^2} \frac{1}{Z} \sum_{\{\vec{s}\}} H^2 e^{-\beta H}$$

Note that $\quad \displaystyle \frac{dZ}{dT} = + \frac{1}{k_B T^2} \sum_{\{\vec{s}\}} H e^{-\beta H}$

So that

$$C = \frac{1}{k_B T^2} \left[\langle H^2 \rangle - \langle H \rangle^2 \right]$$